Exam Calculus 1

November 3, 2014: 9.00-12.00.

This exam has 8 problems. Each problem is worth 1 point. Total: 8 + 1 (free) = 9 points; more details can be found below. Write on each page your name and student number. The use of annotations, books and calculators is not permitted in this examination. All answers must be supported by work. Success.

1. (a) Formulate the principle of mathematical induction.
   (b) The function f is given by $f(x) = x^{-1}$ where $x \neq 0$. The nth derivative of f is denoted by $f^{(n)}(x)$. Prove that for all integers $n \geq 1$,
   $$f^{(n)}(x) \;=\; (-1)^n\, n!\, x^{-(n+1)}$$
   where $n! = 1{\cdot}2{\cdot}3 \cdots n$.

2. Put $\sqrt{i}$ in polar form, find all (complex) solutions z of
   $$\mathrm{e}^{2iz} + \sqrt{i} = 0$$
   and plot them in the complex plane.

3. (a) The function f is defined on some open interval that contains the number a, except possibly at a itself. Give the precise definition of
   $$\lim_{x\to a} f(x) \;=\; L$$
   (b) Prove, using this definition, that
   $$\lim_{x\to 2} \frac{1}{x} \;=\; \frac{1}{2}$$

4. Let f be a function that satisfies the following hypothesis:
   - f is continuous on the closed interval $[a, b]$
   - f is differentiable on the open interval (a, b)

   (a) Formulate the Mean Value Theorem.
   (b) Assume that there exists two numbers m en M such that $m \leq f'(x) \leq M$ for all $x \in (a, b)$ Prove, using the Mean Value Theorem, that
   $$m(b-a) \;\leq\; f(b) - f(a) \;\leq\; M(b-a)$$
   (c) Prove that for all real numbers a en b:
   $$|\sin a - \sin b| \;\leq\; |a - b|$$

5. (a) Use the definition of derivative to show that the derivative of the exponential function $f(x) = a^x$ satsifies $f'(x) = f'(0)\, a^x$ (where $a > 0$).

   (b) The function f is differentiable on $(-\infty, \infty)$. Determine $f(2)$ in the following two cases:

$$\text{(b1)} \int_0^x f(t)dt = x^2(1+x) \qquad \text{(b2)} \int_0^{x^2} f(t)dt = x^2(1+x)$$

6. Evaluate

   (a)

$$\lim_{x\to 0+} x^2 \ln x$$

   (b)

$$\lim_{x\to 0+} x^{x^2}$$

7. Evaluate

   (a)

$$\int (x^2 - 1)\, e^x \, dx$$

   (b)

$$\int_0^e \frac{\ln x}{x}\, dx$$

8. Find the solution $y(x)$ of the initial-value problem

$$x^2 y' + xy = 1 \qquad x > 0 \qquad y(1) = 2$$

Maximum points:

1a	0.4	2	1.0	3a	0.4	4a	0.3	5a	0.5	6a	0.5	7a	0.5	8	1.0
b	0.6			b	0.6	b	0.4	b	0.5	b	0.5	b	0.5		
						c	0.3								